Surface Quotient Singularities and Bigness of the Cotangent Bundle: Part II

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Abstract

We determine a formula for the A_n singularity invariants involved in the Canonical Model Singularities criterion for bigness of the cotangent bundle of a surface of general type appearing in Part I of this work. Our approach of calculating these invariants is of interest on its own. We determine, for a given degree m, the space of symmetric differentials on the complement of the exceptional locus E of the resolution of a germ of an A_n singularity that extend holomorphically across E. We give a full description of the function $\hbar^0(A_n, m)$ giving the codimension of these spaces for each m. We also characterize the precise extent to which the poles along E of the symmetric differentials on the complement are milder than logarithmic poles.

Keywords: A_n singularities, symmetric differentials, big cotangent bundle, extension results, surfaces of general type

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1 Introduction

In Part I of this work [1], we present the (quotient singularities) QS-bigness criterion and its special case, the (canonical model singularities) CMS-criterion, concerning bigness of the cotangent bundle of surfaces of general type X. The CMS-criterion for the bigness of the cotangent bundle of X has two terms: one comes from the topology of the minimal model $X_{\min}$ of X and the other comes from the contribution of the singularities in the canonical model X_{can} of X to the m-asymptotic growth of $h^1(X_{\min}, S^m\Omega^1_{X_{\min}})$. The main purpose of this paper is to determine this contribution when the singularities are of type A_n. Our approach also leads to several extension results for symmetric differentials on the complement of the exceptional locus of the minimal resolution of an A_n singularity.

In Section 1, we give the background and define the strategy to determine the 1st cohomological Ω-asymptotics, $h^1_\Omega(y)$, of a log terminal (quotient) surface singularity y. These are the surface invariants involved in the QS-bigness criterion. Our approach uses the theory developed by Wahl [2], Blache [3] and Langer [4] for the Chern classes (local and global) and the asymptotic Riemann-Roch formulas for orbifold vector bundles.

The 1st cohomological Ω-asymptotics, $h^1_\Omega(y)$, of a surface singularity y is defined by

$$h^1_\Omega(y) = \liminf_{m\to\infty} \frac{h^0(U_y, R^1\sigma_* S^m\Omega^1_{U_y})}{m^3},$$

where U_y is a neighborhood germ of the surface singularity y and $\sigma : \tilde{U}_y \to U_y$ its minimal resolution. It was shown in the proof of the QS-criterion (Theorem 2 of Part I) that if X is the minimal resolution of a surface of general type Y with quotient singularities, then

$$\sum_{y \in \mathrm{Sing}(Y)} h^1_\Omega(y) \le h^1_\Omega(X) := \lim_{m\to\infty} \frac{h^1(X, S^m\Omega^1_X)}{m^3}$$

The right side $h^1_\Omega(X)$ is the measurement of the m-asymptotic growth of $h^1(X, S^m\Omega^1_X)$. The sum on the left side is called *the localized component of* $h^1_\Omega(X)$ and we denote it by $Lh^1_\Omega(X)$. It is a lower bound for $h^1_\Omega(X)$ determined by the singularities of Y.

Our approach to find $h^1_\Omega(y)$ uses the relations between the local invariants appearing in the comparison between the Euler characteristics an orbifold vector bundle V on an orbifold surface Y and of a vector bundle $\tilde{V}$ on its minimal resolution X, $\sigma : X \to Y$, with $V = \sigma_*\tilde{V}$. The key relation for us, requires m-asymptotic results for $V = (S^m\Omega^1_Y)^{\vee\vee}$ (see section 3.1), is:

$$\lim_{m\to\infty} \frac{\hbar^0(y,m) + h^1(y,m)}{m^3} = \frac{1}{3!}\left((c_2(y,T_X) - c_1^2(y,T_X)\right)$$

where $h^1(y,m) := h^0(U_y, R^1\sigma_*S^m\Omega^1_{U_y})$, $c_2(y,T_X)$, $c_1^2(y,T_X) \in \mathbb{Q}$ are the local Chern numbers of y and

$$\hbar^0(y,m) = \dim[H^0(\tilde{U}_y \setminus E, S^m\Omega^1_{\tilde{U}_y})/H^0(\tilde{U}_y, S^m\Omega^1_{\tilde{U}_y})],$$

where $(\tilde{U}_y, E)$ is the minimal resolution of the neighborhood germ (U_y, y).

In Section 2, we present a method to find the invariants $\hbar^0(y,m)$ and their asymptotics when y is an A_n singularity. We show that for fixed n, $\hbar^0(A_n,m)$ is a polynomial of degree 3 in m up to the linear term. In fact, it is a quasi-polynomial in m, i.e. coefficients are periodic functions of m, with the cubic and quadratic coefficients constant. The main interest in this work, with respect to the discussion above, lies in finding the cubic term of such a polynomial. In Theorem 1, we give a closed formula for the cubic and quadratic coefficients of $\hbar^0(A_n,m)$. We note that the special case of A_1 was considered in [5], see [6] for the cubic coefficient and [7] for the quasi-polynomial.

Theorem 1 *Let $(\tilde{X}, E)$ be the minimal resolution of the germ (X,x) of an A_n singularity. Then the following holds for $\hbar^0(A_n,m) = \dim[H^0(\tilde{X}\setminus E, S^m\Omega^1_{\tilde{X}})/H^0(\tilde{X}, S^m\Omega^1_{\tilde{X}})]$:*

(a) $\hbar^0(A_n,m)$ is given by a weighted lattice sum over a polygon $\mathcal{P}_n(m)$:

$$\hbar^0(A_n,m) = \sum_{\substack{\mathbf{x}=(x_1,x_2)\in\mathcal{P}_n(m)\cap\mathbb{Z}^2 \\ x_1+(n+1)x_2 \equiv m \bmod 2}} h_{n,m}(\mathbf{x})$$

where

- *i) the polygon $\mathcal{P}_n(m)$ is symmetric about the x_1-axis and $\mathcal{P}_n(m)\cap\{x_2 \ge 0\}$ is given by the inequalities $x_1 \ge 0$, $x_2 \ge 0$, $x_1 - (n-1)x_2 \le m$ and $-x_1 + (n+1)x_2 \le m+2$.*
- *ii) the weight function $h_{n,m}(\mathbf{x}) = \min\left\{\sum_{r=0}^{n-1} \alpha_{n,m,r}(\mathbf{x}), \beta_{n,m}(\mathbf{x})\right\}$, with*

$$\alpha_{n,m,r}(\mathbf{x}) = \max\left\{0, \frac{m - x_1 + (2r-n+1)x_2}{2}\right\}$$
$$\beta_{n,m}(\mathbf{x}) = \max\{0, m+1-\alpha_{n,m,-1}(\mathbf{x}) - \alpha_{n,m,n}(\mathbf{x})\}.$$

(b) $\hbar^0(A_n,m) = \hbar^0_\Omega(A_n)m^3 + 3\hbar^0_\Omega(A_n)m^2 + O(m)$, with

$$\hbar^0_\Omega(A_n) := \lim_{m\to\infty}\frac{\hbar^0(A_n,m)}{m^3} = \frac{4}{3}\sum_{j=1}^{n}\frac{1}{j^2} - \frac{12n^4+65n^3+117n^2+72n}{6(n+1)^2(n+2)^2}.$$

To understand the weight function, some setup needs to be introduced. Let $\tilde{X}$ be minimal resolution of the A_n affine model $X = \{xz + y^{n+1} = 0\} \subset \mathbb{C}^3$ and E the exceptional locus. Using the smoothing $\pi : \mathbb{C}^2 \to X$ we define a 3-gradation on the algebra $S(\tilde{X} \setminus E) := \bigoplus_{m=0}^{\infty} H^0(\tilde{X} \setminus E, S^m\Omega^1_{\tilde{X}})$. A differential w has 3-degree $(\hat{k}, i, m)$ if $w \in H^0(\tilde{X} \setminus E, S^m\Omega^1_{\tilde{X}})$ and when w is viewed in $H^0(\mathbb{C}^2, S^m\Omega^1_{\mathbb{C}^2})^{\mathbb{Z}_{n+1}}$ it vanishes to order i at 0, for the degree $\hat{k}$ it comes from a block partition on the $\mathbb{Z}_{n+1}-$ invariant differential monomials, see Section 2.2.1.

The weight function $h_{n,m}(\mathbf{x})$ gives the dimension of the obstruction space for differentials in the (x_2, x_1, m)-graded piece of $S(\tilde{X} \setminus E)$ to extend regularly along E. The polygon $\mathcal{P}_n(m)$ lives on the (x_1, x_2)-plane, $x_1 = i$ and $x_2 = \hat{k}$. The weight function naturally induces a decomposition of the polygon $\mathcal{P}_n(m) = \cup_{\ell=1}^{4n} \mathcal{P}_n^{\ell}(m)$ by convex polygons where the weight functions $h_{n,m,\ell}(\mathbf{x}) := h_{n,m}(\mathbf{x})|_{\mathcal{P}_n^{\ell}(m)}$ are defined by a single polynomial of degree 1 in x_1, x_2 (and m).

The function $\hbar^0(A_n, m)$ being a quasi-polynomial of degree 3 in m follows from the theory of polynomial weighted lattice sums over convex polytopes $\mathcal{P}(\mathbf{b})$, where the parameter $\mathbf{b} = (b_1, ..., b_k)$ defines $\mathcal{P}(\mathbf{b})$ via the inequalities, $\mu_l(\mathbf{x}) \leq b_l$, where the linear forms $\mu_l(\mathbf{x})$ are fixed, but $\mathbf{b}$ varies (see [8], [9], [10]). Each polygon $\mathcal{P}_n^{\ell}(m)$ is of the form $\mathcal{P}(\mathbf{b}(m))$ with $\mathbf{b}(m) = (\alpha_{\ell,1}m + \beta_{\ell,1}, ..., \alpha_{\ell,k}m + \beta_{\ell,k}$ with $k = 3, 4$. Relevant to the task of finding the period of the quasi-polynomial $\hbar^0(A_n, m)$ (i.e. the lcm of the periods of the coefficients of the quasi-polynomial) is that $\alpha_{\ell,i}, \beta_{\ell,i}$ and the coefficients of the fixed linear forms $\mu_l(\mathbf{x})$ are in $\mathbb{Q}$ (and we know the denominators).

In future work we will describe non-asymptotic features of the quasi-polynomial $\hbar^0(A_n, m)$. One such feature is a divisibility condition for the period. This result allows us to determine the quasi-polynomials $\hbar^0(A_n, m)$ for low n and use them to investigate the presence of symmetric differentials of low degrees on the resolution of certain classes surfaces (e.g. hypersurfaces in $\mathbb{P}^3$) with A_n type singularities.

In Section 3, we derive applications of our knowledge of $\hbar^0(A_n, m)$ and the theory developed in Section 2 describing the extension properties of differentials on $\tilde{X} \setminus E$. The first application is the main purpose of Part II of this work, that is, finding the closed formula for the 1st cohomological Ω-asymptotics $h^1_{\Omega}(x)$ for A_n singularities.

Theorem 2 *The 1st cohomological Ω-asymptotics of an A_n singularity is given by:*

$$h^1_{\Omega}(A_n) = \frac{n^5 + 19n^4 + 83n^3 + 137n^2 + 80n}{6(n+1)^2(n+2)^2} - \frac{4}{3}\sum_{k=1}^{n} \frac{1}{k^2},$$

In Part I [1] Section 3, Theorem 2 is used in combination with the CMS-bigness criterion to obtain the strongest known results concerning:

i) the minimum $d_{\min}$ of the degrees d for which the deformation equivalence class of a smooth hypersurfaces of $\mathbb{P}^3$ of degree d has representatives with big cotangent bundle. Theorem 4 of Part I shows $d_{\min} \leq 8$, moreover in theory with Theorem 2 and considering resolutions of hypersurfaces with only A_n singularities one could achieve $d_{\min} \leq 6$, but never $d_{\min} = 5$;

ii) bigness of the cotangent bundle of the resolutions of cyclic coverings of $\mathbb{P}^2$ branched along line arrangements, see Theorem 5 of Part I.

The second application is concerned with extension results and characterizes: 1) the precise extent to which the poles along E of symmetric differentials on the complement of the exceptional locus E of the minimal resolution of an A_n singularity are milder than logarithmic poles ([11] 4.14, see also [2] 4.7 showed that the poles are at most logarithmic); 2) a sufficient condition for a symmetric differential on the complement of the exceptional locus E of the minimal resolution of the germ of an A_n singularity to extend holomorphically through E.

Theorem 3 *Let $\sigma : (\tilde{X}, E) \to (X, x)$ and $\pi : (\mathbb{C}^2, 0) \to (X, x)$ be the minimal resolution and the smoothing of the A_n singularity germ (X, x). Then:*

(a) The maximal divisor D such that:

$$H^0(\tilde{X} \setminus E, S^m\Omega^1_{\tilde{X}}) = H^0(\tilde{X}, S^m\Omega^1_{\tilde{X}}(\log E) \otimes \mathcal{O}_{\tilde{X}}(-D))$$

is given by

$$D = \sum_{r=1}^{n} \left(\sum_{j=0}^{\min(r-1,n-r)} \lceil \frac{m-2j}{n+1} \rceil \right) E_r$$

with $E = U^n_{r=1} E_r$ the exceptional locus (ordering such that $E_r \cap E_{r+1} \neq \emptyset$).

(b) Let $w \in H^0(X \setminus x, S^m\Omega^1_X)$ and $\bar{w} \in H^0(\mathbb{C}^2, S^m\Omega^1_{\mathbb{C}^2})$ with $\bar{w}|_{\mathbb{C}^2\setminus 0} = \pi^* w$. Set $ord(w) := \max_i\{\bar{w} \in \mathfrak{m}^i S^m\Omega^1_{\mathbb{C}^2}\}$, $\mathfrak{m}$ the maximal ideal at 0.

Then $\sigma^* w$ extends holomorphically to $\tilde{X}$, if $ord(w) \geq nm$.

2 Background

Let Y be an orbifold surface (i.e. with only quotient singularities) of general type and $\sigma : X \to Y$ its minimal resolution. In this section we succinctly describe the invariants of orbifold surface singularities involved in determining the presence of symmetric differentials on X with an emphasis on the asymptotics $\lim_{m\to\infty} \frac{h^0(X, S^m\Omega^1_X)}{m^3}$ (for more details see Section 1 of Part I of this work [1]).

Riemann-Roch gives:

$$h^0(X, S^m\Omega^1_X) = \int_X \mathrm{ch}(S^m\Omega^1_X)\mathrm{td}(X) + h^1(X, S^m\Omega^1_X) - h^2(X, S^m\Omega^1_X) \tag{1.1}$$

The first term $\int_X \mathrm{ch}(S^m\Omega^1_X)\mathrm{td}(X)$ is a polynomial of degree 3 in m with the coefficients involving the Chern classes of X. The term $h^2(X, S^m\Omega^1_X)$ vanishes for $m \geq 3$ due to Bogomolov's vanishing, [12]. The term $h^1(X, S^m\Omega^1_X)$ has the following decomposition (for further details see proof of Theorem 2 in Part I [1]):

$$h^1(X, S^m\Omega^1_X) = Lh^1(X, S^m\Omega^1_X) + NLh^1(X, S^m\Omega^1_X) \tag{1.2}$$

where the localized component (at the singularities):

$$Lh^1(X, S^m\Omega^1_X) := \sum_{y\in \mathrm{Sing}(Y)} h^1(y, S^m\Omega^1_X) \tag{1.3}$$

with $h^1(y, S^m\Omega^1_X) := h^0(U_y, R^1\sigma_* S^m\Omega^1_{\tilde{X}})$, where U_y is an affine neighborhood with $U_y \cap \mathrm{Sing}(Y) = \{y\}$ and the non-localized component $NLh^1(X, S^m\Omega^1_X) := h^1(Y, \sigma_* S^m\Omega^1_X)$.

The QS-bigness criterion (Theorem 2 of Part I) gives the criterion for the bigness of the cotangent bundle of X ($\lim_{m\to\infty} \frac{h^0(X, S^m\Omega^1_X)}{m^3} \neq 0$):

$$\sum_{y\in \mathrm{Sing}(Y)} \liminf_{m\to\infty} \frac{h^1(y, S^m\Omega^1_X)}{m^3} + \frac{s_2(X)}{3!} > 0 \implies \Omega^1_X \text{ big} \tag{1.4}$$

where $\frac{s_2(X)}{3!} = \frac{c_1^2(X)-c_2(X)}{6}$ is the cubic coefficient and the leading term in m of $\int_X \mathrm{ch}(S^m\Omega_X^1)\mathrm{td}(X)$.

The main goal of this work is to provide a method to find $h^1(y, S^m\Omega_X^1)$ and determine its asymptotics, $\lim_{m\to\infty} \frac{h^1(y,S^m\Omega_X^1)}{m^3}$, when y is an A_n singularity (we will see that this limit exists). We use an indirect approach coming from the theory developed by Wahl [2], Blache [3] and Langer [4] concerning orbifold vector bundles, their Chern classes, and Riemann-Roch formulas (with their asymptotics).

Notation. Hereon, $h^1(y,m) := h^1(y, S^m\Omega_X^1)$ and $h^1(A_n, m) := h^1(y,m)$ where y is the singularity A_n.

For further background on the general theory concerning what follows, see Section 1 of Part I [1] (or see [3], [4], [2]).

The difference between the Euler charateristics of the vector bundle $S^m\Omega_X^1$ on X and of the orbifold vector bundle $\hat{S}^m\Omega_Y^1 := (\sigma_*\Omega_X^1)^{\vee\vee}$ on Y is measured by the sum over all the singular points y of Y of the local Euler characteristic of $\hat{S}^m\Omega_Y^1$ at y:

$$\chi(y,m) := \chi(y, S^m\Omega_X^1) := \hbar^0(y,m) + h^1(y,m) \tag{1.5}$$

$$\hbar^0(y,m) := \dim[H^0(\tilde{U}_y \setminus E, S^m\Omega_X^1)/H^0(\tilde{U}_y, S^m\Omega_X^1)] \tag{1.6}$$

The relations between the Euler characteristic of $\hat{S}^m\Omega_Y^1$ on Y, the orbifold Euler characteristic of $\hat{S}^m\Omega_Y^1$ on Y, and the orbifold Euler characteristic of $S^m\Omega_X^1$ on X, where the orbifold Euler characteristic of an orbifold vector bundle $\mathcal{F}$ on a orbifold Z is $\chi_{\mathrm{orb}}(Z,\mathcal{F}) := \int_Z \mathrm{ch}_{\mathrm{orb}}(\mathcal{F})\mathrm{td}_{\mathrm{orb}}(Z)$ (involving orbifold Chern classes) give:

$$h^1(y,m) = \mu(y,m) - \chi_{\mathrm{orb}}(y,m) - \hbar^0(y,m) \tag{1.7}$$

with:

$$\mu(y,m) := \frac{1}{|G_y|} \sum_{g\in G_y\setminus\{\mathrm{Id}\}} \frac{\mathrm{Tr}(\rho_{\hat{S}^m\Omega_Y^1}(g))}{\det(\mathrm{Id}-g)} \tag{1.8}$$

$$\chi_{\mathrm{orb}}(y,m) := \frac{s_2(y)}{3!}m^3 - \frac{1}{2}c_2(y)m^2 - \frac{c_1^2(y)+3c_2(y)}{12}m + \frac{c_1^2(y)+c_2(y)}{12} \tag{1.9}$$

where:

i) $\mu(y,m)$ is the contribution that the singularity y gives to the discrepancy between the Euler characteristic of $\hat{S}^m\Omega_Y^1$ and the orbifold Euler characteristic of $\hat{S}^m\Omega_Y^1$ on Y. $G_y \subset GL(2,\mathbb{C})$ is the local fundamental group and $\rho_{\hat{S}^m\Omega_Y^1}$ the representation of G_y associated to the orbifold vector bundle $\hat{S}^m\Omega_Y^1$ (see [3] 2.6 for the bijective association of isomorphism classes of representations of G_y to isomorphism classes of germs of orbifold vector bundles at the quotient singularity with local fundamental group G_y).

ii) $\chi_{\mathrm{orb}}(y,m)$ is the contribution that the singularity y gives to the discrepancy between the orbifold Euler characteristics of $S^m\Omega_X^1$ on X and of $\hat{S}^m\Omega_Y^1$ on Y.

$$c_1^2(y) := c_1^2(y, T_X) = c_1^2(y, \Omega_X^1) \in \mathbb{Q},$$

$$c_2(y) := c_2(y, T_X) = c_2(y, \Omega_X^1) \in \mathbb{Q},$$

$$s_2(y) = c_1^2(y) - c_2(y)$$

are local Chern numbers at the singularity y. They are obtained from the local Chern classes $c_i(y, T_X) \in H^{2i}_{\mathrm{dRc}}((\tilde{U}_y, E), \mathbb{Q})$, $\tilde{U}_y = \sigma^{-1}(U_y) \subset X$ and dRc stands for de Rham cohomology with compact support ([3, §3]).

3 The invariants $\hbar^0(y, m)$ and their asymptotics for A_n singularities

3.1 A_n model

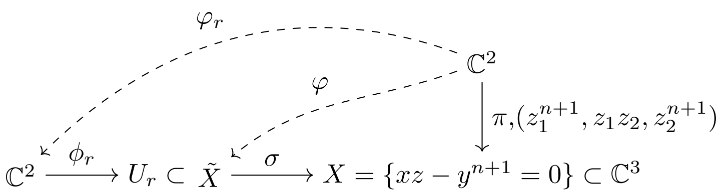
The affine model for the A_n singularity that will be used is $X = \{xz - y^{n+1} = 0\} \subset \mathbb{C}^3$. The affine surface X is the quotient space obtained from $\mathbb{C}^2$ via the diagonal action of $\mathbb{Z}_{n+1}$ coming from the representation $\rho : \mathbb{Z}_{n+1} = \langle \tau \rangle \to SL(2, \mathbb{C})$ with:

$$\rho(\tau) = \begin{bmatrix} \epsilon & 0 \\ 0 & \epsilon^n \end{bmatrix} \tag{2.1}$$

where ϵ is a $n+1$-primitive root of unity.

The standard smoothing of X, $\pi : \mathbb{C}^2 \to X$, is given by $\pi(z_1, z_2) = (z_1^{n+1}, z_1 z_2, z_2^{n+1})$. Let $\sigma : \tilde{X} \to X$ be the minimal good resolution of X, which can be obtained via successive blow ups of the ambient space $\mathbb{C}^3$ at points infinitesimally near the origin (n blow ups for n odd and $n-1$ for n even). Let $\varphi : \mathbb{C}^2 \dashrightarrow \tilde{X}$ be the $n+1$ to 1 rational map $\sigma^{-1} \circ \pi$, whose indeterminacy locus is $\mathrm{Ind}(\varphi) = \{(0,0)\}$.

The resolution $\tilde{X}$ has a covering consisting of $n+1$ open sets U_r, $r = 0, ..., n$, isomorphic to the affine plane. The isomorphisms $\phi_r : \mathbb{C}^2 \to U_r$, with u_1 and u_2 coordinaters for $\mathbb{C}^2$, can be chosen such that the following diagram holds:



and

$$\varphi_r^* u_1 = z_1^{n+1-r} z_2^{-r} \qquad \varphi_r^* u_2 = z_1^{r-n} z_2^{r+1} \tag{2.2}$$

The exceptional locus of σ, $E = E_1 + ... + E_n$, is a sum of n (-2)-curves that intersect transversally with intersection properties given by the A_n-Dykin diagram. The relation between the open covering $\{U_r\}$ with coordinates $u_{i,r} = u_i \circ \phi_r^{-1}$, $i = 1, 2$, and the exceptional set is given by:

$$E_j \subset U_{j-1} \cup U_j \qquad E_j \cap U_r = \emptyset \ \text{ if } \ r \neq j-1, j$$

$$E_j \cap U_{j-1} = \{u_{1,j-1} = 0\} \qquad E_j \cap U_j = \{u_{2,j} = 0\} \tag{2.3}$$

Also relevant is the extension of the exceptional set $\hat{E} = E \cup E_0 \cup E_{n+1}$, where $E_0 = \{u_{2,0} = 0\} \subset U_0$ and $E_{n+1} = \{u_{1,n} = 0\} \subset U_n$. The following holds for all $r = 0, ..., n$:

$$\tilde{X} \setminus \hat{E} = U_r \setminus \{u_{1,r} u_{2,r} = 0\} \tag{2.4}$$

Note: To obtain uniformity in the formulas ahead we also consider $r = -1$ and $r = n+1$, where we define $\varphi_r : \mathbb{C}^2 \to \mathbb{C}^2$ for $r = -1, n+1$ using the same formulas as in (2.2).

The following diagram coming, via restrictions, from diagram (2.2) is also relevant:

$$\begin{array}{ccc} & & \mathbb{C}^* \times \mathbb{C}^* \\ & \swarrow^{\varphi_r} & \big\downarrow \pi \\ \mathbb{C}^* \times \mathbb{C}^* & \xrightarrow{\sigma \circ \phi_r} & X^* := (\mathbb{C}^* \times \mathbb{C}^*)/\mathbb{Z}_{n+1} \end{array} \tag{2.5}$$

3.2 The 3-gradation of the algebra of symmetric differentials $S(\mathbb{C}^* \times \mathbb{C}^*)$

3.2.1 Block partition

The algebra of regular symmetric differentials on $\mathbb{C}^* \times \mathbb{C}^* \subset \mathbb{C}^2$:

$$S(\mathbb{C}^* \times \mathbb{C}^*) = \bigoplus_{m=0}^{\infty} H^0(\mathbb{C}^* \times \mathbb{C}^*, S^m\Omega^1_{\mathbb{C}^2}) = \mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}, dz_1, dz_2]$$

has the natural bi-gradation with bi-graded pieces:

$$S(\mathbb{C}^* \times \mathbb{C}^*)_{(i,m)} = H^0(\mathbb{C}^* \times \mathbb{C}^*, S^m\Omega^1_{\mathbb{C}^2})_{(i)} := \mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}]_{(i)} \otimes \mathbb{C}[dz_1, dz_2]_{(m)}$$

with $i \in \mathbb{Z}$ and $m \in \mathbb{Z}_{\geq 0}$. Also relevant to us is the subalgebra $S(\mathbb{C}^2) = \mathbb{C}[z_1, z_2, dz_1, dz_2]$ of $S(\mathbb{C}^* \times \mathbb{C}^*)$ with induced bi-gradation with bi-graded pieces $H^0(\mathbb{C}^2, S^m\Omega^1_{\mathbb{C}^2})_{(i,m)}$, $i, m \in \mathbb{Z}_{\geq 0}$.

A symmetric differential $w \in S(\mathbb{C}^* \times \mathbb{C}^*)_{(i,m)}$ is said to be bi-homogeneous of *order* i and *degree* m. The symmetric differentials $z_1^{i_1} z_2^{i_2} (dz_1)^{m_1} (dz_2)^{m_2}$ will be called monomials. We have:

$$z_1^{i_1} z_2^{i_2} (dz_1)^{m_1} (dz_2)^{m_2} \text{ is } \mathbb{Z}_{n+1}\text{-invariant} \iff i_1 + m_1 + n(i_2 + m_2) \overset{n+1}{\equiv} 0 \tag{2.6}$$

Denote by $S(\mathbb{C}^* \times \mathbb{C}^*)^{\mathbb{Z}_{n+1}}$ the subalgebra of $\mathbb{Z}_{n+1}$-invariant differentials.

From diagram (2.5) one obtains for each $r = 0, ..., n$:

$$\begin{array}{ccc} & & S(\mathbb{C}^* \times \mathbb{C}^*)^{\mathbb{Z}_{n+1}} \\ & \nearrow^{\varphi_r^*}_{\cong} & \cong \big\uparrow \pi^* \\ S(\mathbb{C}^* \times \mathbb{C}^*) & \underset{\cong}{\xleftarrow{(\sigma \circ \phi_r)^*}} & S(X^*) \end{array} \tag{2.7}$$

The goal is to introduce a 3-gradation of $S(\mathbb{C}^* \times \mathbb{C}^*)$, with graded pieces spanned by symmetric differential monomials, that is respected by the isomorphisms φ_r^*. The structure of the pullback of symmetric differential monomials under the isomorphisms $\varphi_r^* : S(\mathbb{C}^* \times \mathbb{C}^*) \to S(\mathbb{C}^* \times \mathbb{C}^*)^{\mathbb{Z}_{n+1}}$ is the motivation for our choice of 3-gradations appearing below.

The pullback by φ_r of a monomial of degree m, using (2.2), is of the form:

$$\varphi_r^*(u_1^{i_1} u_2^{i_2} (du_1)^{m-q} (du_2)^q) = \sum_{l=0}^{m} c_{ql}(r) z_1^{j_1(i_1,i_2,m,q,r)+l} z_2^{j_2(i_1,i_2,m,q,r)-l} (dz_1)^{m-l} (dz_2)^l \tag{2.8}$$

where

$$\begin{aligned} j_1(i_1, i_2, m, q, r) &= (n+1-r)i_1 + (r-n)i_2 + (n-r)m + (2r-2n-1)q \\ j_2(i_1, i_2, m, q, r) &= (-r)i_1 + (r+1)i_2 + (-r)m + (2r+1)q, \end{aligned} \tag{2.9}$$

and the coefficients $c_{ql}(r)$ are determined by the equality:

$$[(n+1-r)X - rY]^{m-q}\,[(r-n)X + (r+1)Y]^q = \sum_{l=0}^{m} c_{ql}(r) X^{m-l} Y^l. \tag{2.10}$$

(all monomials on the right side of (2.8) are $\mathbb{Z}_{n+1}$-invariant).

The expression (2.8) jointly with (2.9) motivates the partition of the set of all monomials of degree m, say in the coordinates (z_1, z_2), into the following *blocks* of $m+1$ monomials:

$$B_{k,i,m} := \left\{ z_1^{i-k+l} z_2^{k-l} (dz_1)^{m-l} (dz_2)^l) \right\}_{l=0,\ldots,m}, \tag{2.11}$$

where $i, k \in \mathbb{Z}$. The monomials in a single block are either all $\mathbb{Z}_{n+1}$-invariant or all not:

$$B_{k,i,m} \cap S(\mathbb{C}^* \times \mathbb{C}^*)^{\mathbb{Z}_{n+1}} = \begin{cases} B_{k,i,m}, & \text{if } 2k \equiv i+m \bmod n+1 \\ \varnothing, & \text{otherwise.} \end{cases}$$

When dealing with blocks of $\mathbb{Z}_{n+1}$-invariant monomials a distinct indexing will be useful. The $\mathbb{Z}_{n+1}$-invariance condition corresponds to $\hat{k} = \frac{2k-i-m}{n+1} \in \mathbb{Z}$. Set:

$$\hat{B}_{\hat{k},i,m} := B_{k(\hat{k}),i,m} \qquad k(\hat{k}) = \frac{i+m}{2} + \frac{n+1}{2}\hat{k} \tag{2.12}$$

The permissible indices $(\hat{k}, i, m)$ satisfy:

$$(n+1)\hat{k} \equiv i+m \bmod 2 \tag{2.13}$$

Condition (2.13) has a dichotomy:

$$(n+1)\hat{k} \equiv i+m \bmod 2 \iff \begin{cases} i+m \equiv 0 \bmod 2 & n \text{ odd} \\ \hat{k} \equiv i+m \bmod 2 & n \text{ even} \end{cases}$$

3.2.2 3-gradations for $S(\mathbb{C}^* \times \mathbb{C}^*)$ and $S(\mathbb{C}^2)^{\mathbb{Z}_{n+1}}$

Set the 3-gradation of $S(\mathbb{C}^* \times \mathbb{C}^*)$ to be:

$$S(\mathbb{C}^* \times \mathbb{C}^*) = \bigoplus_{\substack{m \in \mathbb{Z}_{\geq 0} \\ i,k \in \mathbb{Z}}} V_{k,i,m} \tag{2.14}$$

where the graded pieces are:

$$V_{k,i,m} := \mathrm{Span}(B_{k,i,m})$$

$(V_{k,i,m}V_{k',i',m'} \subset V_{k+k',i+i',m+m'})$.

We have two variants of a 3-gradation of $S(\mathbb{C}^* \times \mathbb{C}^*)^{\mathbb{Z}_{n+1}}$:

$$S(\mathbb{C}^* \times \mathbb{C}^*)^{\mathbb{Z}_{n+1}} = \bigoplus_{\substack{m\in\mathbb{Z}_{\geq 0} \\ i,k\in\mathbb{Z} \\ 2k-i-m\equiv 0 \text{ mod } n+1}} V_{k,i,m} \tag{2.15}$$

and the other using $\hat{V}_{\hat{k},i,m} := \text{Span}(\hat{B}_{\hat{k},i,m})$ is:

$$S(\mathbb{C}^* \times \mathbb{C}^*)^{\mathbb{Z}_{n+1}} = \bigoplus_{\substack{m\in\mathbb{Z}_{\geq 0} \\ i,\hat{k}\in\mathbb{Z} \\ (n+1)\hat{k}\equiv i+m \text{ mod } 2}} \hat{V}_{\hat{k},i,m} \tag{2.16}$$

For each $r = 0, ..., n$ the pullback morphisms φ_r^* give the isomorphisms:

$$S(\mathbb{C}^* \times \mathbb{C}^*) \xrightarrow{\varphi_r^*} S(\mathbb{C}^* \times \mathbb{C}^*)^{\mathbb{Z}_{n+1}}$$

that by (2.8) respects the 3-gradations. The relations (2.9) describe which graded pieces are sent to which graded pieces:

$$\varphi_r^* V_{k-r\left(\frac{2k-m-i}{n+1}\right),i+(n-2r)\left(\frac{2k-m-i}{n+1}\right),m} = V_{k,i,m}$$

We will be interested in the following reformulation of the above:

$$\varphi_r^* V_{\hat{k},i,m,r} = \hat{V}_{\hat{k},i,m}$$

$$V_{\hat{k},i,m,r} := V_{\frac{i+m}{2}+(\frac{n+1}{2}-r)\hat{k},i+(n-2r)\hat{k},m} \tag{2.17}$$

The above gradation of $S(\mathbb{C}^* \times \mathbb{C}^*)$ induces a 3-gradation on $S(\mathbb{C}^2)$:

$$S(\mathbb{C}^2) = \bigoplus_{\substack{i,m\in\mathbb{Z}_{\geq 0} \\ 0\leq k\leq m+i}} V_{k,i,m}^{\text{reg}} \tag{2.18}$$

The graded pieces are:

$$V_{k,i,m}^{\text{reg}} := V_{k,i,m} \cap H^0(\mathbb{C}^2, S^m\Omega^1_{\mathbb{C}^2}) = \text{Span}(B_{k,i,m}^{\text{reg}})$$

where $B_{k,i,m}^{\text{reg}} := \left\{ z_1^{i_1} z_2^{i_2} dz_1^{m_1} dz_2^{m_2} \in B_{k,i,m} | i_1, i_2 \geq 0 \right\}$.

Remark: The condition $0 \leq k \leq m + i$ is equivalent to $B_{k,i,m}^{\text{reg}} \neq \emptyset$.

We will also use with $j = 1, 2$:

$$B_{k,i,m}^{\text{reg},j} := \left\{ z_1^{i_1} z_2^{i_2} dz_1^{m_1} dz_2^{m_2} \in B_{k,i,m} \mid i_j \geq 0 \right\}$$

For the algebra $S(\mathbb{C}^2)^{\mathbb{Z}_{n+1}}$ we will consider the 3-gradation:

$$S(\mathbb{C}^2)^{\mathbb{Z}_{n+1}} = \bigoplus_{\substack{i,m\in\mathbb{Z}_{\geq 0} \\ |\hat{k}|\leq \frac{i+m}{n+1} \\ (n+1)\hat{k}\equiv i+m \bmod 2}} \hat{V}^{\mathrm{reg}}_{\hat{k},i,m} \tag{2.19}$$

The condition $|\hat{k}| \leq \frac{i+m}{n+1}$ is equivalent to $\hat{B}^{\mathrm{reg}}_{\hat{k},i,m} \neq \emptyset$.

3.3 Finding $\hbar^0(A_n, m)$ and its asymptotics

In this section we derive, for any fixed n, a formula for the function (notation of section 2.1):

$$\hbar^0(A_n, m) = \dim \left(H^0(\tilde{X} \setminus E, S^m\Omega^1_{\tilde{X}}) / H^0(\tilde{X}, S^m\Omega^1_{\tilde{X}}) \right)$$

giving the dimension of the space of obstructions for m-differentials on $\tilde{X} \setminus E$ to extend holomorphically through E. We will observe that $\hbar^0(A_n, m)$ is a quasi-polynomial of degree 3 in m. The main purpose is to determine the leading asymptotics:

$$\hbar^0_\Omega(A_n) = \lim_{m\to\infty} \frac{\hbar^0(A_n, m)}{m^3}$$

3.3.1 From $\tilde{X}$ to $\mathbb{C}^2$

In this section, $\tilde{X}$ is the minimal resolution of the affine model X of the A_n singularity as described in section 2.1. We have the commutative diagram involving the resolution σ and the smoothing π of X:

$$\begin{array}{ccc} & & (\mathbb{C}^2, 0) \\ & \overset{\varphi}{\swarrow} & \downarrow \pi \\ (\tilde{X}, E) & \xrightarrow{\sigma} & (X, x) \end{array}$$

The map φ induces the isomorphisms between the algebras of symmetric differentials:

$$S\left(\tilde{X} \setminus E\right) \xrightarrow[\cong]{\varphi^*} S\left(\mathbb{C}^2\right)^{\mathbb{Z}_{n+1}} \ , \ S\left(\tilde{X} \setminus \hat{E}\right) \xrightarrow[\cong]{\varphi^*} S\left(\mathbb{C}^* \times \mathbb{C}^*\right)^{\mathbb{Z}_{n+1}}$$

(recall: $\hat{E} = E + E_0 + E_{n+1}$ as in 2.1). The first isomorphism follows from

$$H^0\left(\tilde{X} \setminus E, S^m\Omega^1_{\tilde{X}}\right) \xleftarrow[\cong]{\sigma^*} H^0\left(X \setminus \{x\}, S^m\Omega^1_X\right) \xrightarrow[\cong]{\pi^*} H^0\left(\mathbb{C}^2 \setminus \{0\}, S^m\Omega^1_{\mathbb{C}^2}\right)^{\mathbb{Z}_{n+1}}$$

plus the equality

$$H^0\left(\mathbb{C}^2, S^m\Omega^1_{\mathbb{C}^2}\right)^{\mathbb{Z}_{n+1}} = H^0\left(\mathbb{C}^2 \setminus \{0\}, S^m\Omega^1_{\mathbb{C}^2}\right)^{\mathbb{Z}_{n+1}}$$

due to the reflexivity of the sheaf $S^m\Omega^1_{\mathbb{C}^2}$. The second isomorphism follows from similar arguments.

We make use of the following diagram:

$$
\begin{array}{ccccc}
H^0(\tilde{X}\setminus\hat{E}, S^m\Omega^1_{\tilde{X}}) & \xrightarrow[\cong]{\varphi^*} & H^0(\mathbb{C}^*\times\mathbb{C}^*, S^m\Omega^1_{\mathbb{C}^2})^{\mathbb{Z}_{n+1}} & \xleftarrow[\cong]{\varphi_r^*} & H^0(\mathbb{C}^*\times\mathbb{C}^*, S^m\Omega^1_{\mathbb{C}^2}) \\
\uparrow & & \uparrow & \varphi_r^* & \uparrow \\
H^0(\tilde{X}\setminus E, S^m\Omega^1_{\tilde{X}}) & \xrightarrow[\cong]{\varphi^*} & H^0(\mathbb{C}^2, S^m\Omega^1_{\mathbb{C}^2})^{\mathbb{Z}_{n+1}} & & H^0(\mathbb{C}^2, S^m\Omega^1_{\mathbb{C}^2})
\end{array}
$$

to turn the task of finding $\hbar^0(A_n,m)$ from the spaces $\tilde{X}$ and $\tilde{X}\setminus E$ to $\mathbb{C}^2$, via:

$$\hbar^0(A_n,m)=\dim\left(H^0(\mathbb{C}^2,S^m\Omega^1_{\mathbb{C}^2})^{\mathbb{Z}_{n+1}}\Big/\bigcap_{r=0}^{n}\varphi_r^*H^0(\mathbb{C}^2,S^m\Omega^1_{\mathbb{C}^2})\right) \tag{2.20}$$

The above equality holds since a differential $w\in H^0(\tilde{X}\setminus E,S^m\Omega^1_{\tilde{X}})$ has no poles along E_r and E_{r+1} if and only if $\varphi^*w\in\varphi_r^*H^0(\mathbb{C}^2,S^m\Omega^1_{\mathbb{C}^2})$.

Notice that whilst for each r, $\varphi_r^*H^0(\mathbb{C}^2,S^m\Omega^1_{\mathbb{C}^2})\not\subset H^0(\mathbb{C}^2,S^m\Omega^1_{\mathbb{C}^2})$ (differentials will be $\mathbb{Z}_{n+1}-$ *invariant* but not necessarily holomorphic along $\{z_1z_2=0\}$), one has

$$\bigcap_{r=0}^{n}\varphi_r^*H^0(\mathbb{C}^2,S^m\Omega^1_{\mathbb{C}^2})\subset H^0(\mathbb{C}^2,S^m\Omega^1_{\mathbb{C}^2})$$

since $w\in\varphi_0^*H^0(\mathbb{C}^2,S^m\Omega^1_{\mathbb{C}^2})\cap\varphi_n^*H^0(\mathbb{C}^2,S^m\Omega^1_{\mathbb{C}^2})$ is $w=\varphi^*\tilde{w}$ with $\tilde{w}\in H^0(\tilde{X}\setminus E,S^m\Omega^1_{\tilde{X}})$.

3.3.2 3-gradation breakdown of $\hbar^0(A_n,m)$

The 3-gradation breakdown of $\hbar^0(A_n,m)$ follows from the the pullback mappings, φ_r^*, sending graded piece to graded piece of the 3-gradation of $H^0(\mathbb{C}^*\times\mathbb{C}^*,S^m\Omega^1_{\mathbb{C}^2})$ given in (2.14).

Let $w\in H^0(\mathbb{C}^2,S^m\Omega^1_{\mathbb{C}^2})^{\mathbb{Z}_{n+1}}$. Since no monomial belongs to two blocks that generate two distinct graded pieces, the following are equivalent:

i) $(\varphi^*)^{-1}w$ extends regularly along E

ii) $(\varphi^*)^{-1}w_{\hat{k},i,m}$ extends regularly along E, $\forall\ w_{\hat{k},i,m}$ in the $(\hat{k},i,m)$-decomposition of w.

Also relevant to this equivalence and what will follow is the decomposition for each $r=0,...,n$:

$$(\varphi_r^*)^{-1}w_{\hat{k},i,m}=\sum w_{\hat{k},i,m,r}\ ,\qquad w_{\hat{k},i,m,r}\in V_{\hat{k},i,m,r}$$

with this decomposition, the following statements are equivalent to $(\varphi^*)^{-1}w_{\hat{k},i,m}$ extending regularly along E:

a) $w_{\hat{k},i,m,r}\in V^{\text{reg}}_{\hat{k},i,m,r}$, for $r=0,...,n$

b) $w_{\hat{k},i,m,r}\in V^{\text{reg},1}_{\hat{k},i,m,r}$, for $r=-1,...,n$

c) $w_{\hat{k},i,m}\in\bigcap_{r=-1}^{n}\varphi_r^*V^{\text{reg},1}_{\hat{k},i,m,r}$

See the note below (2.4) for the meaning of the case $r=-1$. The artificial case of $r=-1$ in (b) and (c) will be used to streamline formulas. We observe that $\varphi_{r-1}^*V^{\text{reg},1}_{\hat{k},i,m,r-1}=\varphi_r^*V^{\text{reg},2}_{\hat{k},i,m,r}$, for $r=0,...,n+1$.

Proposition 4 *Let $(\tilde{X},E)$ be the germ of the minimal resolution of (X,x) the germ of an A_n singularity. Then:*

$$\hbar^0(A_n,m)=\sum_{\substack{0\le i\le mn-1\\ |\hat{k}|\le\frac{i+m}{n+1}\\ (n+1)\hat{k}\equiv i+m \bmod 2}}\hbar^0(A_n,\hat{k},i,m) \tag{2.21}$$

where $\hbar^0(A_n,\hat{k},i,m) := \dim\left(\frac{\hat{V}^{reg}_{\hat{k},i,m}}{\bigcap_{r=-1}^{n}\varphi_r^*V^{\mathrm{reg},1}_{\hat{k},i,m,r}}\right)$.

Note. The inclusion $\bigcap_{r=-1}^{n}\varphi_r^*V^{\mathrm{reg},1}_{\hat{k},i,m,r} \subset \hat{V}^{\mathrm{reg}}_{\hat{k},i,m}$ holds since $\hat{V}^{\mathrm{reg}}_{\hat{k},i,m} = \varphi_{-1}^*V^{\mathrm{reg},1}_{\hat{k},i,m,-1} \cap \varphi_n^*V^{\mathrm{reg},1}_{\hat{k},i,m,n}$.

Proof The result follows from (2.20), the decomposition:

$$H^0(\mathbb{C}^2,S^m\Omega^1_{\mathbb{C}^2})^{\mathbb{Z}_{n+1}} = \bigoplus_{\substack{i\in\mathbb{Z}_{\geq 0}\\ |\hat{k}|\leq\frac{i+m}{n+1}\\ (n+1)\hat{k}\equiv i+m \bmod 2}} \hat{V}^{\mathrm{reg}}_{\hat{k},i,m}$$

and from the equivalences i) $\iff$ ii) and a) $\iff$ b) $\iff$ c) that imply:

$$\bigcap_{r=0}^{n}\varphi_r^*H^0(\mathbb{C}^2,S^m\Omega^1_{\mathbb{C}^2}) = \bigoplus_{\substack{i\in\mathbb{Z}_{\geq 0}\\ |\hat{k}|\leq\frac{i+m}{n+1}\\ (n+1)\hat{k}\equiv i+m \bmod 2}} \bigcap_{r=-1}^{n}\varphi_r^*V^{\mathrm{reg},1}_{\hat{k},i,m,r}$$

For the upper bound $i\leq mn-1$, see Theorem 3(b). □

3.3.3 Independence of the conditions to have no poles along the E_r and the formula for $h^0(A_n,\hat{k},i,m)$

It follows from Proposition 1 that we need to get a handle on the subspaces $\varphi_r^*V^{\mathrm{reg},1}_{\hat{k},i,m,r} \subset \hat{V}_{\hat{k},i,m}$, $r=-1,...,n$ and determine how they intersect each other.

We will show that the subspaces $\varphi_r^*V^{\mathrm{reg},1}_{\hat{k},i,m,r}$ of $\hat{V}_{\hat{k},i,m}$ are in general position, which implies that the dimension of their intersection is the expected dimension. It also means that the conditions for a given differential not to have poles along the components E_r are independent of each other.

Lemma 1 (Independence of the conditions to have no poles along the E_r) *The subspaces* $\varphi_r^*V^{\mathrm{reg},1}_{\hat{k},i,m,r} \subset \hat{V}_{\hat{k},i,m}$, $r=-1,...,n$, *of* $\hat{V}_{\hat{k},i,m}$ *are in general position.*

Proof Set $\mu_q := z_1^{i_1(\hat{k},i,m)+q}z_2^{i_2(\hat{k},i,m)-q}(dz_1)^{m-q}(dz_2)^q$, $q=0,...,m$, the monomials spanning $\hat{V}_{\hat{k},i,m}$ and $\mu_{q,r} := u_1^{i_1(\hat{k},i,m,r)+q}u_2^{i_2(\hat{k},i,m,r)-q}(du_1)^{m-q}(du_2)^q$, $q=0,...,m$, the monomials spanning $V_{\hat{k},i,m,r}$.

Consider the commutative diagram of isomorphisms:

$$\begin{array}{ccc} V_{\hat{k},i,m,r} & \xrightarrow{\varphi_r^*} & \hat{V}_{\hat{k},i,m} \\ & \searrow^{\Psi_r} & \downarrow{\Psi} \\ & & \mathbb{C}[X,Y]_{(m)} \end{array}$$

where the $\mathbb{C}$-linear map Ψ is defined by $\Psi(\mu_q) = X^{m-q}Y^q$.

From (2.8) and (2.10) of section 2.2, it follows that:

$$\Psi_r(\mu_{r,q}) = [(n+1-r)X - rY]^{m-q}\,[(r-n)X+(r+1)Y]^q$$

and hence the zero locus $Z(\Psi_r(\mu_{r,q})) = (m-q)\,\mathrm{p}_{r,1} + q\,\mathrm{p}_{r,2} \subset \mathbb{P}^1$, where $\mathrm{p}_{r,1} = [r : n+1-r]$ and $\mathrm{p}_{r,2} = [r+1 : r-n]$.

The monomials $\mu_{r,q}$ with no poles along $E_{r+1} \cap U_r = \{u_1 = 0\}$, generating $V^{reg,1}_{\hat{k},i,m,r}$, have $q \geq \max\{0, -i_1(\hat{k}, i, m, r)\}$, hence:

$$\Psi_r(V^{reg,1}_{\hat{k},i,m,r}) = \left\{P \in \mathbb{C}[X,Y] \;\middle|\; Z(P) \geq \max\{0, -i_1(\hat{k}, i, m, r)\}\;\mathrm{p}_{r,2}\right\}$$

Since the points $\mathrm{p}_{r,2}$, $r = -1, ..., n$, are all distinct, the membership conditions for $P \in \Psi_r(V^{reg,1}_{\hat{k},i,m,r})$, $r = -1, ..., n$, are independent. Therefore the subspaces $\Psi_r(V^{reg,1}_{\hat{k},i,m,r}) \subset \mathbb{C}[X,Y]_{(m)}$ are in general position and the result follows.

□

Corollary 1 *For each triple $(\hat{k}, i, m)$, the contribution $\hbar^0(A_n, \hat{k}, i, m)$ to $\hbar^0(A_n, m)$ is:*

$$\hbar^0(A_n, \hat{k}, i, m) = \min\left\{\sum_{r=0}^{n-1} \mathrm{codim}(V^{\mathrm{reg},1}_{\hat{k},i,m,r}, V_{\hat{k},i,m,r})\ ,\ \dim \hat{V}^{\mathrm{reg}}_{\hat{k},i,m}\right\} \tag{2.22}$$

where:

i) $\mathrm{codim}(V^{\mathrm{reg},1}_{\hat{k},i,m,r}, V_{\hat{k},i,m,r}) = \max\left\{0, \frac{m-i}{2} + \frac{2r-n+1}{2}\hat{k}\right\} \leq m$

ii) $\dim \hat{V}^{reg}_{\hat{k},i,m} = \max\left\{0, m+1 - \sum_{r=-1,n} \mathrm{codim}(V^{\mathrm{reg},1}_{\hat{k},i,m,r}, V_{\hat{k},i,m,r})\right\}$

Proof The formula (2.22) follows directly from the subspaces $\varphi_r^* V^{\mathrm{reg},1}_{\hat{k},i,m,r}$ of $\hat{V}_{\hat{k},i,m}$ being in general position (Lemma 1) plus the fact that $\hbar^0(A_n, \hat{k}, i, m) = \mathrm{codim}(\bigcap_{r=-1}^{n} \varphi_r^* V^{\mathrm{reg},1}_{\hat{k},i,m,r}\ ,\ \hat{V}^{\mathrm{reg}}_{\hat{k},i,m})$.

As for the expression in (i), we have that the monomials spanning $V_{\hat{k},i,m,r}$ are $u_1^{i_1+l} u_2^{i_2-l} dz_1^{m-l} dz_2^l$ where $l = 0, ..., m$ and $i_1 = \frac{i-m}{2} + \frac{-2r+n-1}{2}\hat{k}$ (a consequence of (2.17)). Hence, there are $\max\{0, -i_1\}$ monomials which are not regular along $\{u_1 = 0\}$. Note that in the range $|\hat{k}| \leq \lfloor \frac{i+m}{n+1} \rfloor$ we have $\max\{0, -i_1\} \leq m$.

Identity ii) follows from the note after Proposition 1 and Lemma 1. □

3.3.4 The geometry of $h^0(A_n, \hat{k}, i, m)$

In the previous section, the function $\hbar^0(A_n, \hat{k}, i, m)$ was fully determined in Corollary 1. To understand $\hbar^0(A_n, m)$ the geometric properties of the function $\hbar^0(A_n, \hat{k}, i, m)$, when n and m are fixed, play an important role and this section uncovers them.

The function $\hbar^0(A_n, \hat{k}, i, m)$ is an even function relative to $\hat{k}$, a consequence of $\mathrm{codim}(V^{\mathrm{reg},1}_{\hat{k},i,m,r}, V_{\hat{k},i,m,r}) = \mathrm{codim}(V^{\mathrm{reg},1}_{-\hat{k},i,m,n-1-r}, V_{-\hat{k},i,m,n-1-r})$.

Fix n, using corollary 1 we associate to each $m \geq 0$ a polygonal region $\mathcal{P}_n(m)$ in the $(i, \hat{k})-$*plane*. The polygon $\mathcal{P}_n(m)$ is the closure of the region where $\hbar^0(A_n, \hat{k}, i, m)$ is nonvanishing (with i and $\hat{k}$ being considered as continuous parameters). The polygon $\mathcal{P}_n(m)$ has the polygonal decomposition

$$\mathcal{P}_n(m) = \cup_{j=1}^{2n+2} \mathcal{P}_n^j(m)$$

where the $\mathcal{P}_n^j(m)$, $j = 1, ..., 2n+2$, are the regions where $\hbar^0(A_n, \hat{k}, i, m)$ is given by a single affine function on i and $\hat{k}$, making this decomposition unique, see below for more details.

Remark: All the polygons $\mathcal{P}_n^j(m)$ are convex with the exception of two. One can do a convex subdivision of $\mathcal{P}_n(m) = \cup_{j=1}^{2n+2}\mathcal{P}_n^j(m)$ in a natural way,

$$\mathcal{P}_n(m) = \cup_{r=1}^{4n}\mathcal{P'}_n^r(m) \tag{2.23}$$

and more importantly the convex polygons $\mathcal{P'}_n^r(m)$ change with m via
$\mathcal{P'}_n^r(m) = \{x \in \mathbb{R}^2 \mid \mu_{r,l}(x) \leq a_{r,l}m + b_{r,l}, l = 1, ..., k_r, a_{r,l}, b_{r,l} \in \mathbb{Q}\}$ with $\mu_{r,l}$ linear forms with $\mathbb{Z}$-coefficients. This type of decomposition will allow the use of the theory of polynomial weighted lattice sums over convex polytopes (see [8], [9], [10]) to derive properties of the function $\hbar^0(A_n, m)$, such as being a quasi-polynomial of degree 3 in m.

The fact, that there are $2n+2$ polygons $\mathcal{P}_n^j(m)$ follows from the discussion below and the properties of the function $\hbar^0(A_n, \hat{k}, i, m)$ derived from corollary 1. The polygons $\mathcal{P}_n^j(m)$ have some of the properties described in the remark: the coordinates of the vertices are affine functions of m and the slopes of the edges are independent of m.

To determine the polygonal decomposition defined above, we need to consider:

1) The $n+2$ lines coming from the components E_r, $r = 0, ..., n+1$, of $\hat{E}$. These lines are given by $\frac{m-i}{2} + \frac{2(r-1)-n+1}{2}\hat{k} = 0$ and separate the half planes where the $\mathrm{codim}(V^{\mathrm{reg},1}_{\hat{k},i,m,r-1}, V_{\hat{k},i,m,r-1})$ equals either $\frac{m-i}{2} + \frac{2(r-1)-n+1}{2}\hat{k}$ or 0. All the lines pass through the point $(m, 0)$ in the $(i, \hat{k})$-plane. The relevant half plane $\{i \geq 0\}$ is therefore separated in $2(n+2)$ radial sectors with center the point $(m, 0)$.

2) Two extra lines are required to determine the regions where $\dim \hat{V}^{\mathrm{reg}}_{\hat{k},i,m}$ either $= 0$ or $= m+1-\sum_{r=-1,n}\mathrm{codim}(V^{\mathrm{reg},1}_{\hat{k},i,m,r}, V_{\hat{k},i,m,r})$. The two lines of 1) associated with E_0 and E_{n+1} are also required to determine these regions. We have the following:

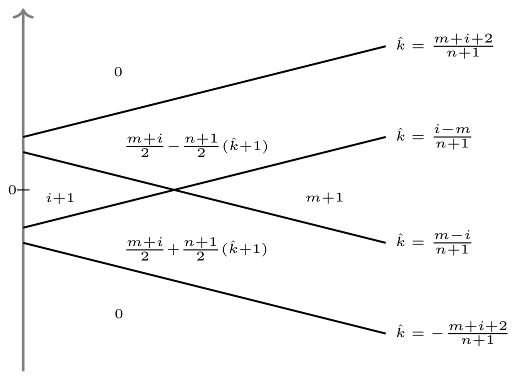


Fig. 1: Graph of $\dim \hat{V}^{\mathrm{reg}}_{\hat{k},i,m}$ on the $(\hat{k}, i)$ plane

3) The $2(n+2)$ sectors defined in 1), except for three between the lines associated to E_n,E_{n+1},E_0 and E_1 and where $\sum_{r=0}^{n-1}\mathrm{codim}(V^{\mathrm{reg},1}_{\hat{k},i,m,r}, V_{\hat{k},i,m,r}) = 0$, each have a line segment separating the regions where $\hbar^0(A_n, \hat{k}, i, m) = \sum_{r=0}^{n-1}\mathrm{codim}(V^{\mathrm{reg},1}_{\hat{k},i,m,r}, V_{\hat{k},i,m,r})$ or $\hbar^0(A_n, \hat{k}, i, m) = \dim \hat{V}^{\mathrm{reg}}_{\hat{k},i,m}$.

4) Altogether there are $2n+2$ polygons $\mathcal{P}_n^j(m)$ where $\hbar^0(A_n, \hat{k}, i, m)$ as a function in i and $\hat{k}$ is defined by a single nontrivial affine expression (the case $n = 3$ is illustrated in the next figure). This follows from 1), 2) and 3) plus the fact that $\dim \hat{V}^{\mathrm{reg}}_{\hat{k},i,m}$ has the same expression in the sectors bounded by the lines E_n, E_{n+1}, E_0 and E_1. Recall that there is the symmetry coming from $\hbar^0(A_n, \hat{k}, i, m)$ being an even function relative to $\hat{k}$.

5) The above are statements are for fixed n and independent of m. As for dependence in m. The coordinates of the vertices of the $2n+2$ polygonal regions are affine functions in m with rational

coefficients. Also important if one wants to give a complete description of the function $\hbar^0(A_n, m)$, the slopes of the boundary lines are independent of m.

6) The affine functions $\hbar^0(A_n, \hat{k}, i, m)_j := \hbar^0(A_n, \hat{k}, i, m)|_{\mathcal{P}^j_n(m)}$ have the following structure for each $j = 1, ..., 2n + 2$,

$$\hbar^0(A_n, \hat{k}, i, m)_j = a_j(n)i + b_j(n)\hat{k} + c_j(n)m + d_j(n) \tag{2.24}$$

$a_j(n), b_j(n), c_j(n), d_j(n) \in \mathbb{Q}$. It is relevant to note the fact that the coefficients in i and $\hat{k}$ are independent of m.

Below we illustrate the function $\hbar^0(A_n, \hat{k}, i, m)$ as a function of $\hat{k}$ and i for $n = 3$. All key features of the general case are present.

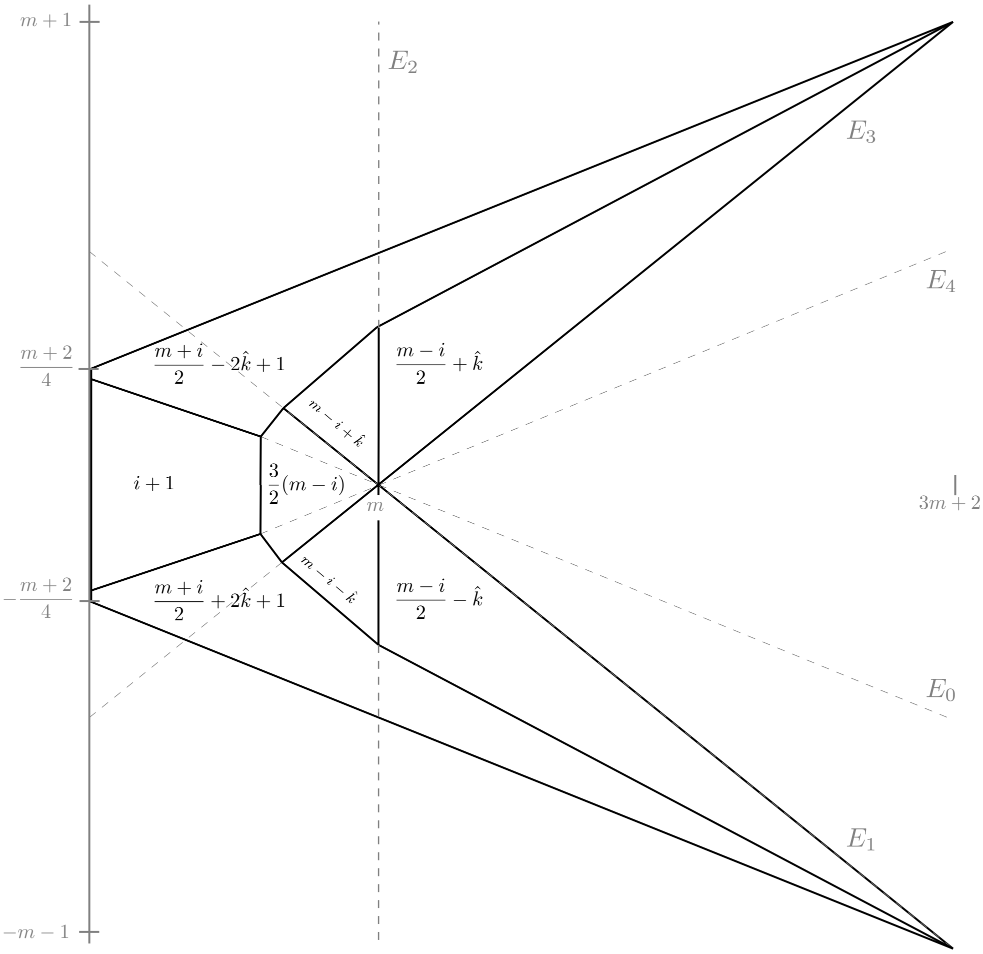


Fig. 2: The function $\hbar^0(A_3, \hat{k}, i, m)$ with m fixed on the $(i, \hat{k})$-plane. Recall that $\hbar^0(A_3, \hat{k}, i, m)$ is only defined at integral points with $4\hat{k} \equiv i + m \bmod 2$.

3.3.5 Closed formula for the asymptotics of $\hbar^0(A_n, m)$

As mentioned in the introduction, one of the aims of this paper is to find the contribution that each A_n singularity gives towards the m-growth asymptotics of $h^1(X, S^m\Omega_X)$ when X is the minimal resolution of a surface of general type Y with canonical singularities. We saw in section 1 that this contribution can be derived from $\hbar^0_\Omega(A_n) = \lim_{m\to\infty} \frac{\hbar^0(A_n,m)}{m^3}$. In this section we give a closed formula for $\hbar^0_\Omega(A_n)$.

Proof (of Theorem 1) Part (a) follows directly from Proposition 4

$$\hbar^0(A_n,m)=\sum_{\substack{0\le i\le mn-1\\ |\hat{k}|\le\frac{i+m}{n+1}\\ (n+1)\hat{k}\equiv i+m \bmod 2}}\hbar^0(A_n,\hat{k},i,m),$$

Corollary 1 describing $\hbar^0(A_n,x_2,x_1,m)(=\ h_{n,m}(x_1,x_2))$, and the polygonal geometry associated to $\hbar^0(A_n,\hat{k},i,m)$ described in 3.3.4.

Part (b) asserts that

$$\hbar^0_\Omega(A_n):=\lim_{m\to\infty}\frac{\hbar^0(A_n,m)}{m^3}=\frac{4}{3}\sum_{j=1}^{n}\frac{1}{j^2}-\frac{12n^4+65n^3+117n^2+72n}{6(n+1)^2(n+2)^2}. \tag{2.25}$$

and

$$\hbar^0(A_n,m)=\hbar^0_\Omega(A_n)m^3+3\hbar^0_\Omega(A_n)m^2+O(m), \tag{2.26}$$

Consider:

$$I(A_n,m):=\iint_{\mathcal{P}_n(m)}h_{n,m}(x)dx$$

Claim 1.

$$\sum_{x\in\mathcal{P}_n(m)\cap\mathbb{Z}^2}h_{n,m}(x)=I(A_n,m)+O(m)$$

If $x=(x_1,x_2)$, set $S_x:=[x_1-\frac{1}{2},x_1+\frac{1}{2}]\times[x_2-\frac{1}{2},x_2+\frac{1}{2}]$. Consider:

$$C_{\mathcal{P}_n(m)}=\{x\in\mathcal{P}_n(m)\cap\mathbb{Z}^2 — S_x\subset\mathcal{P}_n(m)\}$$

$$PC_{\mathcal{P}_n(m)}=\{x\in\mathcal{P}_n(m)\cap\mathbb{Z}^2 — S_x\not\subset\mathcal{P}_n(m)\}$$

$$O_{\mathcal{P}_n(m)}=\{x\in\mathbb{Z}^2\setminus(\mathcal{P}_n(m)\cap\mathbb{Z}^2) — S_x\cap\mathcal{P}_n(m)\neq\emptyset\}$$

Note $\mathcal{P}_n(m)\cap\mathbb{Z}^2=C_{\mathcal{P}_n(m)}\cup PC_{\mathcal{P}_n(m)}$ and:

$$\mathcal{P}_n(m)\subset\cup_{x\in C_{\mathcal{P}_n(m)}\cup PC_{\mathcal{P}_n(m)}\cup O_{\mathcal{P}_n(m)}}S_x \tag{2.27}$$

We start by showing:

$$\sum_{x\in\mathcal{P}_n(m)\cap\mathbb{Z}^2}h_{n,m}(x)=\sum_{x\in\mathcal{P}_n(m)\cap\mathbb{Z}^2}\iint_{S_x}h_{n,m}(y)\,dy+O(m) \tag{2.28}$$

Note: $h_{n,m}(y)=0$ if $y\notin\mathcal{P}_n(m)$.

We need a further decomposition of $C_{\mathcal{P}_n(m)}$ with:

$$SC_{\mathcal{P}_n(m)}=\{x\in C_{\mathcal{P}_n(m)}|\exists j,S_x\subset\mathcal{P}_n^j(m)\}$$

and $MC_{\mathcal{P}_n(m)}=C_{\mathcal{P}_n(m)}\setminus SC_{\mathcal{P}_n(m)}$. The points $x\in C_{\mathcal{P}_n(m)}$, for which S_x is contained in a single polygon $\mathcal{P}_n^j(m)$ and hence where $h_{n,m}(x)$ given by a single affine function on S_x, satisfy:

$$h_{n,m}(x)=\iint_{S_x}h_{n,m}(y)\,dy,\qquad\forall x\in SC_{\mathcal{P}_n(m)} \tag{2.29}$$

If $x\in MC_{\mathcal{P}_n(m)}$ the equality above doesn't hold, but:

$$\left|h_{n,m}(x)-\iint_{S_x}h_{n,m}(y)\,dy\right|<C(n),\qquad\forall x\in MC_{\mathcal{P}_n(m)} \tag{2.30}$$

where $C(n)=\max\{|a_j(n)|,|b_j(n)|;j=1,...,2n+2\}$, with $a_j(n)$ and $b_j(n)$ from (2.24), is a bound on the partial derivatives of $\hbar^0(A_n,y,m)$ (bound independent from m).

Let $x \in PC_{\mathcal{P}_n(m)}$, using the fact that $0 \le h_{n,m}(z) \le 1$ $\forall z \in \partial\mathcal{P}_n(m)$ and $C(n)$, it follows that $\forall y \in S_x$, $|h_{n,m}(y)| < 1 + C(n)$ ($h_{n,m}(y) = 0$ if $y \notin \mathcal{P}_n(m)$). Hence:

$$\left| h_{n,m}(x) - \iint_{S_x} h_{n,m}(y)\, dy \right| < 1 + C(n), \qquad \forall x \in PC_{\mathcal{P}_n(m)} \tag{2.31}$$

The subclaim (2.28) follows from $\#(MC_{\mathcal{P}_n(m)} \cup PC_{\mathcal{P}_n(m)}) \sim O(m)$; it grows linearly with m just as the sum of the lengths of all edges of the polygonal decomposition of $\mathcal{P}_n(m)$.

To prove the claim 1, it remains to show that:

$$\sum_{x \in O_{\mathcal{P}_n(m)}} \iint_{S_x} h_{n,m}(y)\, dy = \iint_{\mathcal{P}_n(m)} h_{n,m}(y)\, dy - \sum_{x \in \mathcal{P}_n(m) \cap \mathbb{Z}^2} \iint_{S_x} h_{n,m}(y)\, dy = O(m).$$

This follows from $|h_{n,m}(y)| < 1 + C(n)$ for all $y \in S_x$, whenever $x \in O_{\mathcal{P}_n(m)}$ and $\#O_{\mathcal{P}_n(m)} = O(m)$.

Claim 2.

$$\sum_{x \in \mathcal{P}_n(m) \cap \mathbb{Z}^2} h_{n,m}(x) = 2\hbar^0(A_n, m) + O(m)$$

Set

$$\hbar^0(A_n, \hat{k}, m) := \sum_{\substack{i=0 \\ (n+1)\hat{k} \equiv i+m \bmod 2}}^{n(m+1)-1} \hbar^0(A_n, \hat{k}, i, m)$$

then $\hbar^0(A_n, m) = \sum_{|\hat{k}| \le m+1} \hbar^0(A_n, \hat{k}, m)$. It is enough to show $\exists\ A(n) > 0$ such that:

$$\left| \sum_{i=0}^{n(m+1)-1} \hbar^0(A_n, \hat{k}, i, m) - 2\hbar^0(A_n, \hat{k}, m) \right| < A(n) \qquad \forall |\hat{k}| \le m+1 \tag{2.32}$$

For fixed $\hat{k}$ (and also n and m), the function $\hbar^0(A_n, \hat{k}, i, m)$ is a piecewise affine function in i with at most $n+2$ affine pieces (and continuous if i is considered a continuous parameter).

One has $\hbar^0(A_n, \hat{k}, i, m) = \frac{1}{2}[\hbar^0(A_n, \hat{k}, i-1, m) + \hbar^0(A_n, \hat{k}, i+1, m)]$ if $i-1, i, i+1$ is the same affine piece. Otherwise one has $|\hbar^0(A_n, \hat{k}, i, m) - \frac{1}{2}[\hbar^0(A_n, \hat{k}, i-1, m) + \hbar^0(A_n, \hat{k}, i+1, m)]| \le C(n)$ with $C(n)$ as above. One also has the boundary condition that

$$0 \le \hbar^0(A_n, \hat{k}, 0, m), \hbar^0(A_n, \hat{k}, n(m+1)-1, m) \le 1.$$

All the above implies $A(n) = 2 + (n+2)C(n)$ works in (2.32), with $2 + C(n)$ bounding the noncancelation at end points and $(n+1)C(n)$ bounding the noncancelation at the (at most) $n+1$ transition points between the affine functions.

Claims 1 and 2 and the symmetry of both $\mathcal{P}_n(m)$ and $h_{n,m}(x)$ with respect to the x_1-axis (i.e. i-axis) imply that

$$\hbar^0(A_n, m) = \iint_{\mathcal{P}_n^+(m)} h_{n,m}(x)\, dy + O(m)$$

where $\mathcal{P}_n^+(m) := \mathcal{P}_n(m) \cap \{x_2 \ge 0\}$. The polygon $\mathcal{P}_n^+(m)$ has a polygonal decomposition $\mathcal{P}_n^+(m) = \cup_{\ell=0}^{n+1} \mathcal{P}_n^\ell(m)$, where for each $\ell = 0, ..., n+1$,

$$h_{n,m,\ell}(x) := h_{n,m}(x)|_{\mathcal{P}_n^\ell(m)}$$

is given by a single affine expression in $x = (x_1, x_2)$ (and m). The polygons $\mathcal{P}_n^\ell(m)$, $\ell = 0, ..., n+1$ will be described via their vertices. To that end, for $j = 1, ..., n$ consider the points:

$$v_j := \left(\frac{2j(m+1)}{j-n-3} + \frac{2(j-1)(m+1)}{-j+n+2} + m, \frac{2(m+1)}{(j-n-3)(j-n-2)} \right)$$

(i) $\mathcal{P}_n^0(m)$:

$$\left(0, \frac{m}{n+1}\right) \to v_1 \to \left(\frac{mn-2}{2+n}, 0\right) \to (0,0) \to \left(0, \frac{m}{n+1}\right),$$

and where $h_{n,m,0}(x) = x_1 + 1$.

(ii) $\mathcal{P}_n^1(m)$:

$$v_1 \to v_2 \to (m,0) \to \left(\frac{mn-2}{2+n}, 0\right) \to v_1,$$

and $h_{n,m,1}(x) = \frac{n(m-x_1)}{2}$.

(iii) $\mathcal{P}_n^j(m)$, $j = 2, \dots, n$:

For $j = 2, \dots, n-1$,

$$v_j \to v_{j+1} \to (m,0) \to v_j$$

and for $j = n$,

$$v_n \to (n(m+1)-1, m+1) \to (m,0) \to v_n$$

and the function $h_{n,m,j}(x) = \frac{1}{2}((j-1)-n)(x_1-(j-1)x_2-m)$.

(iv) $\mathcal{P}_n^{n+1}(m)$:

$$\left(0, \frac{m+2}{n+1}\right) \to (n(m+1)-1, m+1) \to v_n \to \cdots$$
$$\cdots \to v_2 \to v_1 \to \left(0, \frac{m}{n+1}\right) \to \left(0, \frac{m+2}{n+1}\right)$$

and where $h_{n,m,n+1}(x) = \frac{1}{2}(x_1 - x_2(n+1) + m + 2)$.

The above $\mathcal{P}_n^\ell(m)$ and $h_{n,m,\ell}(x)$, $\ell = 0, ..., n+1$ give:

$$\hbar^0(A_n, m) = \sum_{\ell=0}^{n+1} \iint_{\mathcal{P}_n^\ell(m)} h_{n,m,\ell}(x)dx = \hbar_\Omega^0(A_n)m^3 + 3\hbar_\Omega^0(A_n)m^2 + O(m)$$

with $\hbar_\Omega^0(A_n)$ as in (2.25). □

Corollary 2 *The invariants $\hbar_\Omega^0(A_n)$ increase with n and are bounded. Moreover:*

$$\lim_{n\to\infty} \hbar_\Omega^0(A_n) = \frac{2\pi^2}{9} - 2$$

Remark:

i) The invariant $\hbar_\Omega^0(A_1)$ was known [6] (appeared in [5] unfortunately with an error). The A_1 case is quite direct since the exceptional locus has a single component. Some cases of $\hbar_\Omega^0(A_n)$ for low n were known to the authors [13] and [14].

ii) The function $\hbar^0(A_n, m)$ for fixed n is a quasi-polynomial in m of degree 3. This follows from the theory of polynomial weighted lattice sums over convex polytopes $\mathcal{P}(\mathbf{b})$, $\mathbf{b} = (b_1, ..., b_k)$, defined by k inequalities, $\mu_l(x) \le b_l$, where the linear forms $\mu_l(x)$ are fixed, but $\mathbf{b}$ varies ([8], [9], [10]). A natural convex polygon decomposition of the polygons $\mathcal{P}_n^+(m)$ with the required properties can be found by decomposing $\mathcal{P}_n^{n+1}(m)$ into n polygons by introducing vertical line segments above the points v_2,...,v_n.

iii) In Theorem 1, we showed that the cubic and quadratic coefficients of $\hbar^0(A_n, m)$ have period 1. In future work, we describe a divisibility condition for the the least common multiple of the periods of the coefficients of the quasi-polynomial for all n, which allows us to determine $\hbar^0(A_n, m)$ for low n. In the case of A_1 the least common multiple of the periods is 6 [7]. Knowing the functions $\hbar^0(A_n, m)$ can be used to obtain information on the degrees of symmetric differentials that occur on a surface which is a resolution of a surface with A_n singularities.

Example 1 For A_2 singularities the lcm of the periods of the coefficients is also 6 and $\hbar^0(A_2, m)$ is described by the polynomials:

$$\hbar^0(A_2,m) = \begin{cases} \frac{29}{216}m^3 + \frac{29}{72}m^2 + \frac{1}{12}m & m \equiv 0 \pmod 6 \\ \frac{29}{216}m^3 + \frac{29}{72}m^2 + \frac{1}{8}m - \frac{143}{216} & m \equiv 1 \pmod 6 \\ \frac{29}{216}m^3 + \frac{29}{72}m^2 + \frac{7}{36}m - \frac{2}{27} & m \equiv 2 \pmod 6 \\ \frac{29}{216}m^3 + \frac{29}{72}m^2 + \frac{1}{8}m + \frac{3}{8} & m \equiv 3 \pmod 6 \\ \frac{29}{216}m^3 + \frac{29}{72}m^2 + \frac{1}{12}m - \frac{10}{27} & m \equiv 4 \pmod 6 \\ \frac{29}{216}m^3 + \frac{29}{72}m^2 + \frac{17}{72}m - \frac{7}{216} & m \equiv 5 \pmod 6 \end{cases}$$

4 Applications

4.1 The 1st cohomological Ω-asymptotics of A_n singularities

Let X be the minimal resolution of an orbifold surface Y of general type. The asymptotics of the localized component $Lh^1(X, S^m\Omega^1_X)$ of $h^1(X, S^m\Omega^1_X)$ described in (1.3) plays a role in the QS-bigness criterion (1.4).

In this section, we establish the formula for the contribution given by each A_n singularity to $Lh^1(X, S^m\Omega^1_X)$. This contribution consists of:

$$h^1_\Omega(A_n) := \lim_{m\to\infty} \frac{h^1(A_n, m)}{m^3}$$

and it is called the 1−cohomological Ω-asymptotics of A_n (the limit exists).

Proof (of Theorem 2) Using relation (1.7) for A_n singularities,

$$h^1(A_n,m) = \mu(A_n,m) - \chi_{\text{orb}}(A_n,m) - \hbar^0(A_n,m)$$

we find $h^1_\Omega(A_n)$ to be

$$h^1_\Omega(A_n) = \lim_{m\to\infty} \frac{\mu(A_n,m)}{m^3} - \lim_{m\to\infty} \frac{\chi_{\text{orb}}(A_n,m)}{m^3} - \hbar^0_\Omega(A_n).$$

The invariant $\hbar^0_\Omega(A_n)$ was determined in Theorem 1. The invariants $\chi_{\text{orb}}(A_n, m)$ are given by formula (1.9) along with the local Chern numbers

$$c_1^2(A_n) = 0 \qquad \text{and} \qquad c_2(A_n) = e(E) - \frac{1}{|G_{A_n}|} = \frac{n(n+2)}{n+1},$$

where $e(E)$ is the topological Euler characteristic of the exceptional locus of the minimal resolution and $|G_{A_n}|$ is the order of the local fundamental group of the A_n singularity ([3] 3.18). Hence:

$$\lim_{m\to\infty} \frac{\chi_{\text{orb}}(A_n,m)}{m^3} = \frac{s_2(A_n)}{3!} = -\frac{n(n+2)}{6(n+1)}.$$

Now, we turn our attention to the invariants $\mu(A_n, m)$. The key feature of these invariants is that

$$\lim_{m\to\infty} \frac{\mu(A_n,m)}{m^3} = 0$$

This is shown in [3] 4.4 or [4]) (it follows from general results on reflexive sheaves on quotient singularities, see also [2]). A complete description of the invariants $\mu(A_n, m)$ is known to the authors and will appear in future work. For example, for fixed n, $\mu(A_n, m)$ is a quasi-polynomial of degree 1 in m. □

Corollary 3 *The invariants $h^1_\Omega(A_n)$ increase with n and:*

$$\lim_{n\to\infty} h^1_\Omega(A_n) = \infty$$

Remark: The QS-bigness criterion (1.4) has $h^1_\Omega(A_n)$ as the contribution of A_n to the localized component $Lh^1(X, S^m\Omega^1_X)$ of $h^1(X, S^m\Omega^1_X)$, while in the bigness criterion in [15] the contribution of A_n to $h^1(X, S^m\Omega^1_X)$ is $\frac{1}{2}\lim_{m\to\infty}\frac{\chi(A_n,m)}{m^3}$. It follows from Theorem 1, Corollary 2, Theorem 2 and Corollary 3 that the contribution of each A_n in the QS-bigness criterion is always larger and approaches twice the contribution in the criterion of [15] as $n \to \infty$. In Part I of this work, we present the implications of this remark towards the wider range of pair of Chern numbers for which the CMS-bigness criterion can hold when compared to the criterion in [15].

Table 1: Computed values of $h^1_\Omega(A_n)$ for low n

n	1	2	3	4	5	6	7
$h^1_\Omega(A_n)$	$\frac{4}{27}$	$\frac{67}{216}$	$\frac{1283}{2700}$	$\frac{577}{900}$	$\frac{106819}{132300}$	$\frac{1030727}{1058400}$	$\frac{5431459}{4762800}$

4.2 Extension of symmetric differentials for A_n singularities

Let $(\tilde{X}, E)$ be the germ of the minimal resolution of a quotient singularity (X, x). It is well known that symmetric differentials $w \in H^0(\tilde{X}\setminus E, S^m\Omega^1_{\tilde{X}})$ acquire mild poles along the components of the exceptional divisor E. More precisely, the poles of $w \in H^0(\tilde{X}\setminus E, S^m\Omega^1_{\tilde{X}})$ along E are at worst logarithmic, i.e. $H^0(\tilde{X}\setminus E, S^m\Omega^1_{\tilde{X}}) = H^0(\tilde{X}, S^m\Omega^1_{\tilde{X}}(\log E))$ ([11] 4.14, see also [2] 4.7).

We show that for A_n singularities, the poles that $w \in H^0(\tilde{X}\setminus E, S^m\Omega^1_{\tilde{X}})$ can acquire at the exceptional locus are milder (and to what extent) than logarithmic poles (see [6] for A_1). More precisely, in Theorem 3, we give the maximal effective divisor D such that

$$H^0(\tilde{X}\setminus E, S^m\Omega^1_{\tilde{X}}) = H^0(\tilde{X}, S^m\Omega^1_{\tilde{X}}(\log E)\otimes\mathcal{O}_{\tilde{X}}(-D))$$

.

4.2.1 Decomposition and the order of differentials in $S(\tilde{X}\setminus E)$

In this section, $\tilde{X}$ is the minimal resolution of the affine model X of the A_n singularity as described in section 2.1. We have the commutative diagram involving the resolution σ and the smoothing π of X:

$$\begin{array}{ccc} & & (\mathbb{C}^2, 0) \\ & \overset{\varphi}{\swarrow} & \downarrow \pi \\ (\tilde{X}, E) & \xrightarrow{\sigma} & (X, x) \end{array}$$

The map φ induces the isomorphisms between the algebras of symmetric differentials (see section 2.3.2):

$$S\left(\tilde{X}\setminus E\right) \underset{\cong}{\xrightarrow{\varphi^*}} S\left(\mathbb{C}^2\right)^{\mathbb{Z}_{n+1}}$$

$$S\left(\tilde{X}\setminus\hat{E}\right)\xrightarrow[\cong]{\varphi^*}S\left(\mathbb{C}^*\times\mathbb{C}^*\right)^{\mathbb{Z}_{n+1}} \tag{3.1}$$

(recall that $\hat{E}=E+E_0+E_{n+1}$ as in 2.1).

Using the isomorphism (3.1) and the 3-gradation of $S\left(\mathbb{C}^*\times\mathbb{C}^*\right)^{\mathbb{Z}_{n+1}}$ described in 2.2.2, we obtain the $(\hat{k},i,m)-$decomposition of differentials $w\in S\left(\tilde{X}\setminus\hat{E}\right)$:

$$w=\sum_{\substack{m\in\mathbb{Z}_{\geq 0}\\ i,\hat{k}\in\mathbb{Z}}} w_{\hat{k},i,m} \tag{3.2}$$

where the $w_{\hat{k},i,m}\in H^0(\tilde{X}\setminus\hat{E},S^m\Omega^1_{\tilde{X}})$ are such that $\varphi^*w_{\hat{k},i,m}\in\hat{V}_{\hat{k},i,m}$. If $w\in S\left(\tilde{X}\setminus E\right)$, then we have

$$w=\sum_{\substack{m,i\in\mathbb{Z}_{\geq 0}\\ |\hat{k}|\leq\frac{i+m}{n+1}}} w_{\hat{k},i,m}. \tag{3.3}$$

In this case, we additionally have that $\varphi^*w\in S(\mathbb{C}^2)$ and hence $\varphi^*w_{\hat{k},i,m}\in\hat{V}^{\text{reg}}_{\hat{k},i,m}$.

We say a differential $w\in S\left(\tilde{X}\setminus\hat{E}\right)$ is of *type* $(\hat{k},i,m)$ if $\varphi^*w\in\hat{V}_{\hat{k},i,m}$. The expression (3.2) is the decomposition of w relative to the $(\hat{k},i,m)$-types.

We define the *order* of a symmetric differential $w\in S\left(\tilde{X}\setminus\hat{E}\right)$ to be:

$$\operatorname{ord}(w)=\min\left\{i\;\middle|\;\varphi^*w=\sum_{i,\hat{k}\in\mathbb{Z}}w_{\hat{k},i,m}\text{ and }w_{\hat{k},i,m}\neq 0\right\}, \tag{3.4}$$

i.e. $\operatorname{ord}(w)$ is the smallest order of the differential monomials appearing in the monomial decomposition of φ^*w. The $\operatorname{ord}(w)$ can also be described as the order of vanishing of φ^*w at $0\in\mathbb{C}^2$, this is consistent with the definition of order in Theorem 3(a)) for $w\in H^0(X\setminus x,S^m\Omega^1_X)$.

4.2.2 Comparison with logarithmic poles

We characterize the allowed poles of a differential $w\in H^0(\tilde{X}\setminus E,S^m\Omega^1_{\tilde{X}})$ along E via a comparison to the maximum poles allowed on logarithmic symmetric differentials $\mu\in H^0(\tilde{X}\setminus E,S^m\Omega^1_{\tilde{X}}(\log E))$.

Proof (of Theorem 3(a)) Let $w\in H^0(\tilde{X}\setminus E,S^m\Omega^1_{\tilde{X}})$. The worst pole of w along a component E_r of E will be the worst pole attained by one of its $(\hat{k},i,m)$-type components in the decomposition (3.3).

Let w be of $(\hat{k},i,m)$-type. To determine the poles of w along the components of E, we examine w on the coordinate patches U_r, $r=0,...,n$, (described in 2.1) covering $\tilde{X}$. Set $w_r:=w|_{U_r}$, note $w_r=(\varphi_r^*)^{-1}(\varphi^*w)$ hence in $(\varphi_r^*)^{-1}\hat{V}_{\hat{k},i,m}$. Using 2.17, it follows that w_r is in the span of the monomials in the block $B_{\frac{i+m}{2}+(\frac{n+1}{2}-r)\hat{k},i+(n-2r)\hat{k},m}$.

The relevant observation towards the comparison of the poles of $w\in H^0(\tilde{X}\setminus E,S^m\Omega^1_{\tilde{X}})$ to logarithmic poles is the fact that we can rewrite the monomials in a block $B_{k,i,m}$, in the following form:

$$B_{k,i,m}=\left\{z_1^{i+m-k}z_2^k\frac{dz_1^{m-q}}{z_1^{m-q}}\frac{dz_2^q}{z_2^q}\right\}_{q=0,\dots,m}$$

Hence w_r (of type $(\hat{k},i,m)$) is given by a sum of the following monomials:

$$u_1^{\frac{i+m}{2}+(\frac{n-1}{2}-r)\hat{k}} u_2^{\frac{i+m}{2}+(\frac{n+1}{2}-r)\hat{k}} \frac{du_1^{m-q}}{u_1^{m-q}} \frac{du_2^q}{u_2^q}, \qquad q = 0, ..., m \tag{3.5}$$

Next we prove the result for A_n with $n \geq 2$ (the case $n = 1$ follows from the same argument after a minor setup adjustment). For $r = 1, ..., n-1$:

$$S^m\Omega^1_{\tilde{X}}(\log E)|_{U_r} = \bigoplus_{q=0}^{m} \mathcal{O}_{U_r} \frac{du_1^{m-q}}{u_1^{m-q}} \frac{du_2^q}{u_2^q}$$

(recall: for each $r = 1, ..., n-1$, $E_r \cap U_r = \{u_2 = 0\}$ and $E_{r+1} \cap U_r = \{u_1 = 0\}$).

It follows from (3.5) that the order of the poles along E_r of the monomials involved in the sum giving w_r deviates from the highest pole order possible for a logarithmic symmetric differential by subtracting $\frac{i+m}{2} + (\frac{n+1}{2} - r)\hat{k}$. Hence:

$$w_r \in H^0\left(U_r, S^m\Omega^1_{\tilde{X}}(\log E) \otimes \mathcal{O}(-D)\right)$$

with

$$D = \left[\frac{i+m}{2} + \left(\frac{n+1}{2} - r\right)\hat{k}\right] E_r + \left[\frac{i+m}{2} + \left(\frac{n-1}{2} - r\right)\hat{k}\right] E_{r+1}$$

An easy consequence of the above is:

$$H^0(\tilde{X} \setminus E, S^m\Omega^1_{\tilde{X}}) = H^0(\tilde{X}, S^m\Omega^1_{\tilde{X}}(\log E))$$

This follows since the condition that $D \geq 0$ in all $r = 1, ..., n-1$ is equivalent to $|\hat{k}| \leq \frac{i+m}{n-1}$, which holds since the $(\hat{k}, i, m)$-components of $w \in H^0(\tilde{X} \setminus E, S^m\Omega^1_{\tilde{X}})$ satisfy $|\hat{k}| \leq \frac{i+m}{n+1}$.

The full strength of the result follows from:

$$a_r = \min\left\{\tfrac{i+m}{2} + (\tfrac{n+1}{2} - r)\hat{k} \;\middle|\; i \geq 0, |\hat{k}| \leq \tfrac{i+m}{n+1}, (n+1)\hat{k} \equiv i + m \bmod 2\right\}$$

satisfies:

$$a_r := \begin{cases} \sum_{j=1}^{r} \lceil \frac{m+2-2j}{n+1} \rceil & 1 \leq r \leq \lfloor \frac{n+1}{2} \rfloor \\ \sum_{j=1}^{n+1-r} \lceil \frac{m+2-2j}{n+1} \rceil & \lfloor \frac{n+1}{2} \rfloor < r \leq n \end{cases}$$

Set $D = \sum_{r=1}^{n} a_r E_r$. By construction all the $(\hat{k}, i, m)$-components of w, and hence w, belong to $H^0(\tilde{X}, S^m\Omega^1_{\tilde{X}}(\log E) \otimes \mathcal{O}_{\tilde{X}}(-D))$ and the maximality of D is guaranteed by the definition of a_r and the fact. □

4.2.3 Order and holomorphic extension

Part (b) of Theorem 3 gives a criterion for the holomorphic extension to $\tilde{X}$ of symmetric differentials on $\tilde{X} \setminus E$ involving the order of the differentials. This result is used through out section 2, it makes explicit that the $h^0(A_n, m)$ are finite since it implies that the polygon $\mathcal{P}_n(m)$ over which the lattice weighted sum giving $h^0(A_n, m)$ is bounded. We note that this result can also be derived from corollary 1.

Proof (of Theorem 3 part (b)) If $w \in H^0(\tilde{X} \setminus E, S^m\Omega^1_{\tilde{X}})$, then the $(\hat{k}, i, m)$-decomposition of w has $w = \sum w_{\hat{k},i,m}$, with $i \in \mathbb{Z}_{\geq 0}$ and $|\hat{k}| \leq \frac{i+m}{n+1}$. The differential w extends regularly to $\tilde{X}$ if all the $w_{\hat{k},i,m}$ do. Note that each $w_{\hat{k},i,m} \in H^0(\tilde{X} \setminus E, S^m\Omega^1_{\tilde{X}})$, hence no need to check about poles along E_0 or E_{n+1}.

The conditions to guarantee no poles of $w_{\hat{k},i,m}$ along E_r, $r = 1, ..., n$ are obtained using (3.5). From (3.5) it follows that all w of type $(\hat{k}, i, m)$ do not acquire a pole along E_r if and only if:

$$\frac{i+m}{2} + \left(\frac{n+1}{2} - r\right)\hat{k} \geq m \tag{3.6}$$

First observation is that the r conditions (3.6) on the triples $(\hat{k}, i, m)$ can be reduced to the conditions for $r = 1$ and $r = n$. Second observation is that for $i < m$ the conditions can't hold. Finally, if $i \geq m$, the conditions become:

$$|\hat{k}| \leq \frac{i - m}{n - 1}$$

The condition above holds for the possible $\hat{k}$, i.e. in the range given by $|\hat{k}| \leq \frac{i+m}{n+1}$, if $i \geq nm$.

□

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